LIE GROUPS AND LIE ALGEBRAS FINAL EXAMINATION

Total marks: 100

(1) Define a solvable Lie algebra, and a nilpotent Lie algebra. Give an example of a nilpotent Lie algebra, and an example of a solvable Lie algebra which is not nilpotent (justify your answers). State the theorems of Engel and Lie. Prove that a finite dimensional complex Lie algebra is nilpotent, if and only if, every two dimensional subalgebra of it is abelian. (4+6+6+8 = 24 marks)

(2) Define a simple Lie algebra, and a semisimple Lie algebra. Give an example of a simple Lie algebra, and an example of a semisimple Lie algebra which is not simple (justify your answers). Show that a Lie algebra is semisimple, if and only if, it has no nonzero abelian ideals. (4+6+8 = 18 marks)

(3) Describe all finite dimensional irreducible representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Prove that the Heisenberg Lie algebra over $\mathbb{C}$ has no finite dimensional faithful irreducible representations. (8+8 = 16 marks)

(4) State the two criteria of Cartan. Show that the Killing form on $\mathfrak{gl}(n, \mathbb{C})$ is given by $\kappa(a, b) = 2nTr(ab) - 2Tr(a)Tr(b)$ for $a, b \in \mathfrak{gl}(n, \mathbb{C})$. Using this or otherwise, prove that $\mathfrak{sl}(n, \mathbb{C})$ is semisimple for $n \geq 2$. (6+10+10 = 26 marks)

(5) Let L be any finite dimensional Lie algebra. Prove that the image $ad(L)$ of L under the ad map is an ideal in the Lie algebra $Der(L)$ of derivations of L. Prove that $ad(L) = Der(L)$, if L is a finite dimensional complex semisimple Lie algebra. (6+10 = 16 marks)

Date: October 29, 2014.